Bayesian Method: A Natural Tool for Processing Geotechnical Information

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6.1 Introduction

In geotechnical analysis and prediction, it is a common practice for engineers to consider data from multiple sources. In many occasions, the combination of information is often made based on engineering judgement and experience. From a mathematical perspective, the Bayesian method has been widely used to combine information from multiple sources for the purpose of updating prior knowledge given new information. The potential application of Bayesian methods in geotechnical engineering has been explored by many researchers. In practice, however, the application of Bayesian methods is still quite limited. The objective of this report is to illustrate the benefits of using Bayesian methods for processing geotechnical information. The structure of this report is as follows. First, the need for information combination in geotechnical engineering is analyzed, followed by a brief introduction about the general theory and computational techniques for Bayesian computations. Then, several typical examples are presented to show how the Bayesian method can be used to tackle different geotechnical problems. Finally, the potential application of the Bayesian method in the Eurocode is explored, and the possible limitations of the Bayesian method are discussed. We hope that this report can provide a useful guide for practicing engineers to identify the merits of solving geotechnical problems using a Bayesian perspective, and thus encourage them to take advantage of the Bayesian method in practice when applicable. The computational details, however, are not the emphasis of this report.

6.2 Need for information combination in geotechnical engineering

Unlike other science and engineering disciplines, the practice of geotechnical engineering is always amenable to various uncertainties resulting from insufficient site investigation data, time and budget constraint, local experience, expected natural hazards, unpredictable environmental and social impact, construction disturbance, imperfect design models, etc. From the perspective of geotechnical design, uncertainties can be categorized into parameter uncertainty and model uncertainty. Each source of uncertainty can potentially result in unsatisfactory geotechnical performance with associated casualty and economic loss, so it is necessary to explicitly evaluate and quantify parameter uncertainty and model uncertainty in the context of geotechnical reliability-based design, including the commonly-known load and resistance factor design (LRFD).

Given an established design method in a design code, a realistic challenge for practicing geotechnical engineers lies in how to determine design values for soil parameters. This challenge is caused by several factors including the uncertain depositional process and soil history and the limited number of boreholes that the clients can afford to characterize the soil profile and determine the design soil parameter values. To this end, either factor-of-safety-based design using conservatively estimated parameters or probabilistic analysis can be used. In either of the two approaches, the estimated design soil parameters are based on the limited and existing information gained through the initial geological survey, in-situ borehole drilling and laboratory testing, and local experience. Existing knowledge could lead to prior information that may not represent the actual soil conditions and there is a practical need to update the soil parameters using new information, such as the observed responses in the field (Peck 1969).

Variability in soil properties stems from various sources of uncertainty that can be grouped into two categories: aleatory uncertainty and epistemic uncertainty. The word aleatory is evolved from the Latin alea, which means the rolling of dice, and the aleatoric uncertainty refers to the intrinsic randomness of a phenomenon. In geotechnical engineering, the aleatory uncertainty includes spatial variability and random testing errors. The word epistemic is evolved from the Greek episteme (i.e., knowledge). The epistemic uncertainty is caused by lack of knowledge or data. In geotechnical engineering, the epistemic uncertainty is related to measurement procedures and limited data availability (Whitman 1996).

Both aleatory uncertainty and epistemic uncertainty can be addressed using combined prior information and newly observed information in geotechnical engineering. For example, the aleatoric uncertainty that is reflected in a random field of a spatially varying soil parameter can be calibrated given sufficient field load tests. Given more data from case histories, the knowledge on soil parameters, the epistemic uncertainty, can be back-calculated as posterior information. As more observational data are introduced, the uncertainties can be reduced through updating the mean values and decreasing the variance of each parameter, which can benefit the subsequent stages of design and construction. Admittedly, the amount of data required depends on the number of uncertain variables. There is always a trade-off between the site investigation effect and the improved knowledge on design soil parameters. It is noted that the site investigation cost can be optimized using approaches such as the Bayesian method. Depending on the specific problem, the parameter uncertainty, or model uncertainty or both can be updated through back analysis.

6.3 Bayesian method: concept and computational techniques

The uncertainties in the geotechnical design can be modelled using random variables. Those random variables are represented with probability distributions that quantify the knowledge in the model parameters and the model itself. The Bayesian approach provides a rigorous framework to reduce uncertainty as the performance of the geotechnical design is known. Let vector θ denote uncertain variables to be updated with the observation data **D**. The Bayesian method can be applied to both continuous variables and discrete random variables. As an example, suppose the elements of θ are all continuous variables, and the prior knowledge about θ can be denoted by a probability density function (PDF), $f(\theta)$. Let's denote $L(\theta|\mathbf{D})$ as a likelihood function, indicating the chance to observe **D** given θ . One may refer to Juang et al. (2015) on how to construct the likelihood function using various types of geotechnical data. Based on Bayes' theorem, the prior knowledge about θ and the knowledge learned from the observed data can be combined as follows:

$$f(\boldsymbol{\theta} \mid \mathbf{D}) = \frac{L(\boldsymbol{\theta} \mid \mathbf{D}) f(\boldsymbol{\theta})}{\int \cdots \int \int L(\boldsymbol{\theta} \mid \mathbf{D}) f(\boldsymbol{\theta}) \mathrm{d}\boldsymbol{\theta}} \tag{6-1.}$$

where $f(\theta|\mathbf{D})$ = posterior PDF of θ representing the combined knowledge. In Eq. (6-1), both the likelihood function $L(\theta|\mathbf{D})$ and prior PDF $f(\theta)$ affect $f(\theta|\mathbf{D})$, and the role of the likelihood function will become more dominant as the amount of observed data increases.

Based on the law of total variance, it can be shown that the following inequality holds:

$$\mathrm{Var}(\boldsymbol{\theta}) \geq \mathrm{E}_{\boldsymbol{\theta}}\left(\mathrm{Var}(\boldsymbol{\theta} \mid \mathbf{d})\right) \tag{6-2.}$$

where $\mathrm{Var}(\theta)$ denotes the prior variance of θ, $\mathrm{E}_{\theta}(\mathrm{Var}(\theta|\mathbf{D}))$ denotes the average value of the posterior variance of θ. Eq. (6-2) indicates that the posterior variance is *on average smaller* than the prior variance, by an amount that depends on the variation in the posterior means over the distribution of the possible data (Gelman et al. 2013). This inequality implies that if one consistently uses the Bayesian method, uncertainty reduction can be eventually achieved in the long run. However, there is no guarantee that the Bayesian method can always achieve uncertainty reduction in each individual application. The uncertainty in the posterior distribution may be larger than that in the prior distribution if the observed data significantly contradicts with the prior knowledge.

The posterior distribution $f(\boldsymbol{\theta}|\mathbf{D})$ in Eq. (6-1) is generally difficult to evaluate except for some special cases, such as the case where conjugate priors can be employed. In recent years, along with the advances in computational statistics, many methods have become practical for calculating $f(\boldsymbol{\theta}|\mathbf{D})$. Several commonly used computational methods will be described in the next section.

6.3.1 Conjugate prior

If the posterior distribution $f(\boldsymbol{\theta}|\mathbf{D})$ is in the same family as the prior probability distribution $f(\boldsymbol{\theta})$, the prior and posterior are then called conjugate distributions, and the prior is called a conjugate prior for the likelihood function. When the conjugate prior is adopted, the posterior distribution can be obtained analytically, and thus greatly simplifying the computational work involved. A comprehensive summary of conjugate priors can be found in the literature such as Ang and Tang (2007), Gelman et al. (2013), and Givens and Hoetings (2013).

6.3.2 Direct integration method

Based on the definition of the mean and the covariance matrix, the posterior statistics of $\boldsymbol{\theta}$ can be evaluated using the following equations:

$$\mu_{i|\mathbf{D}} = \int \theta_i f\left(\theta_i \mid \mathbf{D}\right) \mathrm{d}\theta_i \tag{6-3.}$$

$$\sigma_{i|\mathbf{D}}^2 = \int \left(\theta_i - \mu_{i|\mathbf{D}}\right)^2 f\left(\theta_i \mid \mathbf{D}\right) \mathrm{d}\theta_i \tag{6-4.}$$

where $f(\theta_i|\mathbf{D})$ = posterior PDF of the ith element of $\boldsymbol{\theta}$, $\mu_{i|\mathbf{D}}$ = posterior mean of θ_i, and $\sigma_{i|\mathbf{D}}$ = posterior standard deviation of θ_i, respectively.

In principle, all existing methods such as Gaussian quadrature (e.g., Christian & Baecher 1999) for integration can be used to evaluate the above integrals. However, the computational work involved with the direction integration method may increase significantly with the dimension of $\boldsymbol{\theta}$. Thus, the direct integration method is often used for low dimensional problems.

6.3.3 Markov Chain Monte Carlo (MCMC) simulation

Albeit the apparent simplicity of Eq. (6-1), obtaining meaningful statistical information might require high dimensional integration which could be computationally very challenging. A common approach to evaluate the posterior distribution is to use sampling methods such as Monte Carlo (MC) and Monte Carlo Markov Chains (MCMC) methods (Brooks et al. 2011, Liu 2004, Robert and Casella 2004). The basic idea of MCMC simulation is to draw samples from a target distribution iteratively by means of a Markov chain that converges to the target distribution. When the Markov chain reaches its equilibrium state, the samples from the Markov chain are also those of the posterior distribution. Thus, these samples can also be used for inferring the posterior properties of the target distribution, and for subsequent geotechnical reliability analysis. In recent years, MCMC simulation has been increasingly used in geotechnical engineering.

6.3.4 System identification (SI) method

In many cases, the parameters of a geotechnical model can hardly be determined with accuracy. The system identification (SI) method (e.g., Tarantola 2005), through embedding a deterministic model into the Bayesian formulation, is specifically developed for updating the parameters of a mechanical model based on observed data. Let $g(\boldsymbol{\theta})$ be a geotechnical model with $\boldsymbol{\theta}$ denoting uncertain input parameters in this model. Let $\mathbf{D}$ denote the observed system response. Assume the prior knowledge about $\boldsymbol{\theta}$ can be denoted by a multivariate normal distribution with a mean of $\boldsymbol{\mu}_{\boldsymbol{\theta}}$ and a covariance matrix of $\mathbf{C}_{\boldsymbol{\theta}}$. Assume that the observational uncertainty can be described by a multivariate normal distribution with a mean vector of zero and a covariance matrix of $\mathbf{C}_{\mathbf{D}}$. Assume further that the model uncertainty can be described by a multivariate normal distribution with a mean vector of zero and a covariance matrix of $\mathbf{C}_m$. With the above assumptions, $\boldsymbol{\theta}^*$ can be found by maximizing the posterior density function, or equivalently, minimizing the following misfit function (Tarantola 2005)

$$2S(\boldsymbol{\theta}) = \left[g(\boldsymbol{\theta}) - \mathbf{D}\right]^T \mathbf{C}_T^{-1} \left[g(\boldsymbol{\theta}) - \mathbf{D}\right] + \left(\boldsymbol{\theta} - \boldsymbol{\mu}_{\boldsymbol{\theta}}\right)^T \mathbf{C}_{\boldsymbol{\theta}}^{-1} \left(\boldsymbol{\theta} - \boldsymbol{\mu}_{\boldsymbol{\theta}}\right) \tag{6-5.}$$

where $\mathbf{C}_T = \mathbf{C}_{\mathbf{D}} + \mathbf{C}_m$.

Let $\boldsymbol{\theta}^*$ be the point where the misfit function is minimized. Based on the system Identification method, the posterior density function $f(\boldsymbol{\theta}|\mathbf{D})$ is approximated by a multivariate normal distribution with a mean of $\boldsymbol{\mu}_{\boldsymbol{\theta}|\mathbf{D}}$ and a covariance matrix of $\mathbf{C}_{\boldsymbol{\theta}|\mathbf{D}}$, where $\boldsymbol{\mu}_{\boldsymbol{\theta}|\mathbf{D}}$ and $\mathbf{C}_{\boldsymbol{\theta}|\mathbf{D}}$ are defined as (Tarantola 2005):

$$\boldsymbol{\mu}_{\boldsymbol{\theta}|\mathbf{D}} = \boldsymbol{\theta}^* \tag{6-6.}$$

$$\mathbf{C}_{\boldsymbol{\theta}|\mathbf{D}} = \left(\mathbf{G}^{\mathrm{T}}\mathbf{C}_T^{-1}\mathbf{G} + \mathbf{C}_{\boldsymbol{\theta}}^{-1}\right)^{-1} \tag{6-7.}$$

$$\mathbf{G} = \frac{\partial g(\boldsymbol{\theta}^*)}{\partial \boldsymbol{\theta}} \tag{6-8.}$$

In Eq. (6-5), the effect of modeling uncertainty and observational uncertainty on the posterior distribution are represented by $\mathbf{C}_\theta$ and $\mathbf{C}_\mathbf{m}$, and their overall effect are jointly represented by $\mathbf{C}_\mathbf{T}$. If the model uncertainty is significantly larger than the observational uncertainty, $\mathbf{C}_\mathbf{T}$ will be dominated by $\mathbf{C}_\mathbf{m}$, and in such a case the posterior distribution will be more affected by the model uncertainty. Similarly, if the observational uncertainty is significantly larger than the model uncertainty, the posterior distribution will be more affected by the observational uncertainty. Usually, the posterior distribution will be more affected by the prior distribution when the model and observational uncertainties increase. One can refer to Tarantola (2005) for a detailed discussion on the interplay between modeling, observational uncertainty, and prior information as well as their effect on the posterior distribution.

6.3.5 Other methods

In geotechnical engineering, several other techniques are also used for evaluating the posterior PDF, such as the extended Bayesian method (e.g., Honjo et al. 1994) and the first order second moment Bayesian method (Gilbert 1999), and the importance sampling or the Latin Hypercube Sampling (Choi et al. 2006). Bayesian computational statistics is a field that is experiencing rapid progress. The interested readers may refer to the literature such as Gelman et al. (2013) for greater details about recent computational techniques for estimating posterior distributions. The Stochastic Finite Element Method (SFEM) includes a set of techniques which enable the propagation of parameter uncertainty through a deterministic Finite Element Method (FEM) (Ghanem & Spanos, 1991, Le Maître & Knio, 2010, Sudret 2008). As the SFEM and the Bayesian approach regard the model parameters as random variables, both can be used seamlessly. There has been some development in this direction (El Moselhy & Marzouk, 2012 and Cañavate et al. 2015).

6.4 Examples

The practical use of the Bayesian method is illustrated using several geotechnical design and analysis applications involving slope stability, foundation design and testing, and liquefaction analyses. The examples will demonstrate the advantages and limitations of the Bayesian approach for solving practical design problems in geotechnical engineering.

6.4.1. Back-analysis of soil properties from failed slopes

Slope stability problems are generally associated with uncertainties in the estimation of pore pressure regimes, soil/rock properties, and geometry of the failure surface. For failed slopes, Bayesian probabilistic approaches allow for utilizing information on the location and geometry of the observed failure surface to update prior probability distributions of the shear strength parameters (ϕ' and c') and the pore pressure regime (r_u). The major strength of probabilistic back-analysis techniques for slopes is the recognition that there are numerous combinations of parameters that could result in the slope failure and the ability to quantify the relative likelihoods of these combinations. Zhang et al. (2010) presented an approach that uses a minimization procedure of a misfit function to refine/update the distributions of ϕ', c', and r_u. The main assumption is that all the parameters are normally distributed.

The example targets a case history involving the stability of a proposed highway in Algeria (Hasan and Najjar 2013), as shown in Figure 6-1. The prior mean values of ϕ' and c' were estimated as 21^o and 15 kPa, respectively. By combining uncertainties due to spatial variability and statistical uncertainties, coefficients of variation (COVs) of 0.31 and 0.55 were determined for ϕ' and c', respectively. The assumed prior pore water pressure regime was modelled with a mean r_u of 0.33 and an associated COV of 0.50. The high COV of 0.50 reflects the lack of site-specific piezometers.

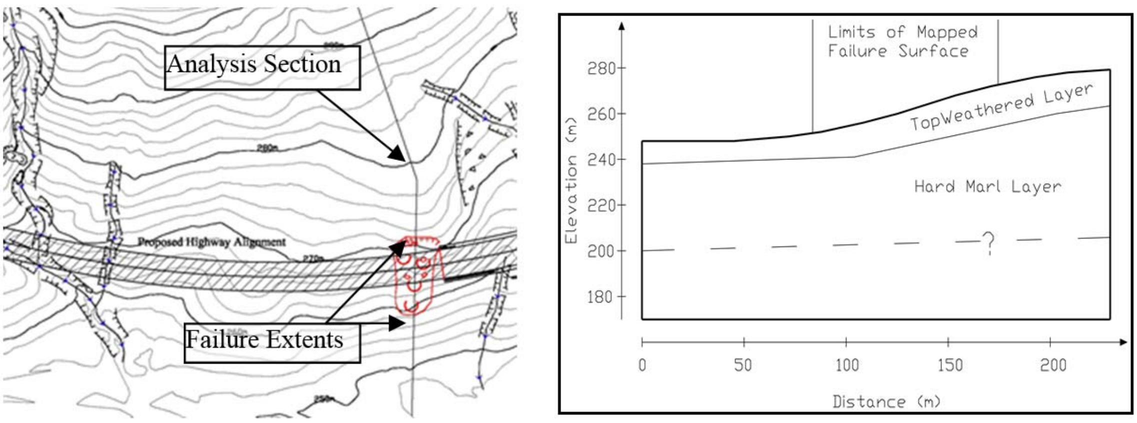

Figure 6-1 Plan and Section View for highway alignment and failure zone (Hasan and Najjar 2013).

Results of the updating process assuming statistically independent parameters indicate reductions in the updated mean ϕ' (from 21 to 17.3 degrees) and c' (from 15 to 12.1 kPa) and an increase in the mean r_u (from 0.33 to 0.42). These results are expected since the assumed prior mean values corresponded to a safety factor of ~ 1.5. Reductions in ϕ' and c' along with an increase in r_u were required for failure conditions to prevail. Results also indicate that the standard deviations decreased for the updated case (5.5%, 27%, and 21.2% reductions in the standard deviations of c', ϕ' and r_u). Finally, results indicated that although the prior parameters were assumed to be statistically independent, the updated parameters were found to be correlated with the following correlation coefficients: $\rho_{c,ø}$=-0.32, $\rho_{ø,ru}$=0.78, and $\rho_{c,\ ru}$ = 0.29.

To investigate the sensitivity of the results to the correlation between c' and ø', the back analysis was repeated for $\rho_{c,ø}$ values of -0.25, -0.5, -0.75, and -0.95. Results indicate that the assumption of negative correlation between c' and ø' results in an appreciable increase in the updated mean values of c', ø', and r_u with the effect being stronger as the correlation is assumed to be stronger (see Hasan and Najjar 2013). On the other hand, the updated standard deviations were found to be less sensitive to the assumed correlation coefficient.

Since uncertainties exist in the failure surface, it is important to determine whether that affects the results of the updating process. Results where the failure plane is varied indicate that the updated mean values of c' and ø' are sensitive to the assumed failure surface, indicating that accurate mapping of the failure surface is required for an accurate estimation of the soil properties.

6.4.2. Shallow foundation reliability based on spatially variable soil data

This example demonstrates the potential and benefits of using Bayesian analysis for incorporating information from spatially distributed data on soil into a reliability analysis. It is taken from (Papaioannou and Straub 2016). Exemplarily, the reliability of a centrically loaded rigid strip footing embedded in silty soil is evaluated. The bearing capacity of the foundation depends on the friction angle of the silty soil, which is a spatially variable property. To identify the friction angle, direct shear tests of soil probes, taken at different depths in the area of the foundation are performed. The measurement outcomes are used to learn the spatial distribution of the soil property through application of Bayesian statistical analysis.

In reliability assessment, spatially variable properties are typically modeled by random variables with reduced variance to account for the spatial averaging effect. In this example, it is demonstrated that such an approach can be extended to a case in which spatial data are used to learn the distribution of spatially variable properties within a Bayesian context. This simplified random variable model is compared to a random field model that explicitly represents the spatial variability of the soil property, and provides the most accurate solution at the highest modeling and computational cost. One important detail: in the random variable model, the data are used to learn the statistics of the soil property and cannot be used to learn its posterior auto-covariance function. Therefore, spatial averaging is performed with the prior auto-correlation function.

Figure 6-2 illustrates the influence of the prior scale of fluctuation of the random field on the prior and posterior reliability estimates obtained with the RV (random variable) approach with spatial averaging and the RF (random field) approach. In the left panel of Figure 6-2, one can observe that the prior reliability estimates calculated with the two approaches agree well. However this is not the case for the posterior reliability estimates shown in the right panel of Figure 6-2. In the RF approach, the reliability increases fast when the scale of fluctuation becomes large. This is because the area of influence of the measurements increases as the prior spatial correlation increases. At low scales of fluctuation, the posterior statistics become almost uniform along the depth of the failure surface and the reliability increases again due to spatial averaging according to the prior correlation structure. The results obtained with the RV approach assume that the posterior correlation is the same as the prior independent of the prior scale of fluctuation. At low scales of fluctuation, this assumption is valid and the reliability estimates are close to the ones obtained with the RF approach. However, with increasing scale of fluctuation, the assumption that the spatial variability of the posterior is not influenced by the measurements is unrealistic and the reliability is significantly underestimated.

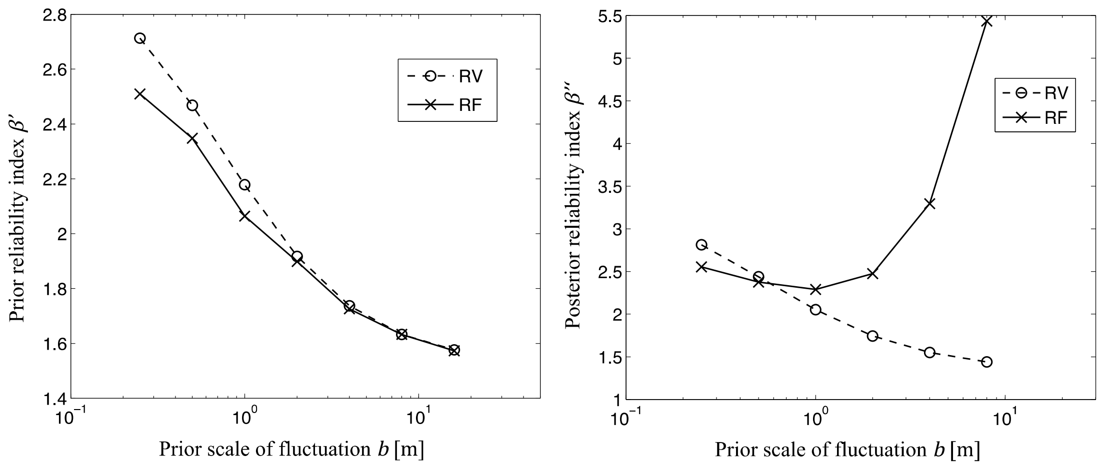

Figure 6-2 Prior and posterior reliability index vs. scale of fluctuation for the two modeling approaches: The random variable (RV) approach and random field (RF) approach.

6.4.3. Updating pile capacity at a site with load test results

In the last three decades, several efforts have targeted analyzing the impact of pile load tests on the design of foundations in the framework of a reliability analysis. Examples include the work of Baecher and Rackwitz (1982), Zhang and Tang (2002), Zhang (2004), Najjar and Gilbert (2009), Kwak et al. (2010), Park et al. (2011, 2012), Abdallah et al. (2015a,b), and Huang et al. (2016). In these studies, results of pile load tests are used to update the mean, median, lower bound, or actual capacity distributions of piles using Bayesian techniques. In this section, an example is presented to illustrate how the updating process is implemented. In the example, the uncertainty in the pile capacity is modeled by a conventional lognormal distribution with (1) a coefficient of variation δ_r that represents the uncertainty due to spatial variability in pile capacity in a given site and (2) an uncertain mean capacity that incorporates the model uncertainty of the pile capacity prediction method.

The mean of the pile capacity (r_{mean}) at a given site is generally a random variable. The mean and COV of r_{mean} are typically estimated from databases of pile load tests from the ratio of measured to predicted capacities, λ. As an example, Zhang (2004) reports COV values ranging from 0.21 to 0.57 for about 14 methods of pile capacity prediction. The COV of the pile capacity δ_r reflects the uncertainty due to within-site spatial variability (0.1 to 0.2) and is generally not updated in the analysis. In this example, the mean and COV of λ are assumed to be 1.30 and 0.50, respectively and δ_r is assumed to be equal to 0.2. Zhang (2004) presents the Bayesian formulation that allows for updating the pile capacity distribution for cases involving (1) pile load tests that are conducted to failure, and (2) proof load tests. The main difference is the form of the likelihood function.

For cases where piles carry the proof load, the mean value of the updated pile capacity increases with the number of tests while the COV value decreases. This is expected to result in increases in the reliability of the updated pile design at the site. For cases where some of the tested piles fail at carrying the proof load, the updated mean of the pile capacity decreases significantly as the number of positive tests decreases. For such cases the updated reliability could be lower than the prior.

Figure 6-3 shows the reliability index β for single piles designed with a factor of safety (FOS) of 2.0 and verified by several proof tests. The β value for the prior case is 1.49. If one positive proof test is conducted, then β will be updated to 2.23. The updated β will continue to increase if more positive proof tests are conducted. In the cases when not all tests are positive, the reliability of the piles will decrease with the number of tests that are not positive. The shaded zone in Figure 6-3 indicates that target reliability levels could be achieved with 1 to 3 positive tests with proof levels of twice the design load, provided that the piles act as part of a system. For a structure supported by four or fewer piles where the system effect may not be dependable, the target reliability should be slightly larger (ex. β = 3.0) than those presented in Figure 6-3. Zhang (2004) shows that one or two successful proof load tests that are conducted at three times the design load are required to achieve a target reliability index of 3.0, as necessitated by non-redundant piles. Based on the above-mentioned methodology, a rational decision making framework (Najjar et al. 2016a,b) could be envisaged to facilitate the choice of a load test program that has the maximum expected benefit to the project.

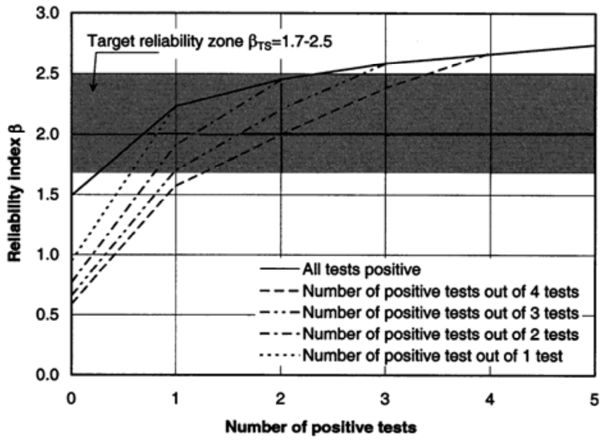

Figure 6-3 Variation of the reliability index with proof load test results (Zhang 2004).

6.4.4. Back-Analysis to determine the undrained strength of liquefied soil

Lateral spreading is the finite, lateral movement of gently to steeply sloping, saturated soil deposits caused by earthquake-induced liquefaction (Kramer 2016). A key parameter controlling the extent of lateral movements is the undrained strength of the liquefied ground. In this example we present a case history where a well-documented lateral spread that occurred during the M8.0 Pisco Earthquake is used along with Bayesian Updating to estimate the undrained strength of the liquefied ground. The lateral spread occurred near the community of Canchamaná, on a marine terrace where pervasive liquefaction was observed (GEER 2007). Post-earthquake investigations as well as a comparison of pre- and post-earthquake satellite images were used to obtain a detailed displacement field over the entire 3km by 1 km where the lateral spread was observed (Cox et al. 2010). In addition, a comprehensive field characterization study that included a series of Standard Penetration Tests (SPT) that covered the complete extent of the marine terrace was completed about 2.5 years after the earthquake. Details of this case history are presented in Gangrade et al. (2015).

The displacement field across the Canchamaná area was non-uniform, with displacements concentrated along certain cross sections. On the other hand, the SPT measurements had a more or less uniform distribution throughout the terrace. This is compatible with the distribution of liquefaction features over the area. Slope stability analysis indicated that the difference in displacements were likely due to differences in surface topography, which in turn imposed different driving forces on the liquefied soil. A key parameter in the stability analyses was the undrained strength of the liquefied soil. Current methodologies, such as Olson and Stark (2002), correlate this strength to corrected SPT N values:

$$\frac{S_u}{\sigma_v'} = mN_{1,60} + c \tag{6-9.}$$

where *c* and *m* are fitting parameters. The relationship applies for SPT≤12; a different slope is applied for SPT>12 (Davies and Campanella 1994).

For the purpose of this study, we assume that *m* is given and we let *c* be a random variable. The objective of the Bayesian Updating analysis is to determine the value of *c* in Equation (9) that is compatible with the observations of failure/no-failure at four cross-sections in the Canchamaná complex. We select a uniform prior for *c* within the bounds given by Olson and Stark (2002). The posterior distribution for *c*, given that some bins have failed and some have not failed, is given by:

$$f(c \mid F_i\bar{F}_j) = \frac{P(F_i \mid c)P(\bar{F}_{ji} \mid c)}{P(F_i\bar{F}_j)} f(c) \tag{6-10.}$$

where $f(c)$ is the prior distribution, F_i denotes *failure* of bin *i*, and $\bar{F}_j$ denotes *no-failure* in bin *j*. The conditional failure probabilities are computed from pseudo-static slope stability analyses assuming a lognormal distribution for the input PGA. Posterior distributions for *c* were obtained for pair-wise combinations of a failed and no-failed bin. Modal values of *c* from these distributions are plotted in Figure 6-4 for different values of median and standard deviation of the PGA. Recorded values of PGA at nearby stations, as well as estimated intra-event standard deviation (σ_{lnPGA}) from existing ground motion prediction equations can be used to obtain best-estimate values of the parameter *c*. In this example, the use of Bayesian Updating allowed for a formal way of incorporating previous knowledge (e.g., the Olson and Stark model) with observations to improve the predictive ability of the model. The analyses could be improved through more rigorous determination of the prior distributions and a more formal inclusion of model uncertainties (see discussion in Section 6.3).

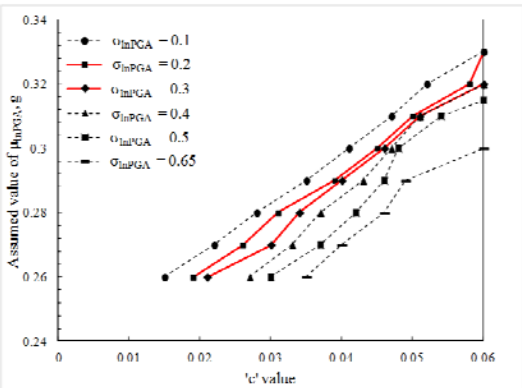

Figure 6-4 Correlation between 'c' values and the estimated mean PGA (μ_{lnPGA}).

6.4.5. Identification of underground soil stratification

Identification of underground soil stratification is an important aspect in geotechnical site characterization, even within a single type of soil. Consider, for example, London Clay Formation (LCF). It is well recognized that the LCF contains five depositional cycles (i.e. lithological units) and each cycle records an initial marine transgression followed by gradual shallowing of the sea. Because geotechnical properties (e.g. strength, stiffness and consolidation characteristics) of London Clay vary significantly in different lithological units or soil strata, it is of practical significance to identify the soil strata in London Clay so that the geotechnical property data within the same soil strata can be compared or used effectively.

Bayesian model class selection (BMCS) method has been developed to properly identify underground soil strata (Wang et al. 2014&2016b). In BMCS, a model class is referred to a family of stratification models that share the same number of soil strata but have different model parameters (Cao and Wang 2013, Wang et al. 2013). The number of soil strata is considered as a variable k, which is a positive integer varying from 1 to a maximum possible number N_{Lmax}. Therefore, there are N_{Lmax} candidate model classes M_k, $k = 1, 2, \dots, N_{Lmax}$, and the k-th model class M_k has k soil strata. For a given set of site-specific observation data $\underline{X}_M$, the plausibility of each model class is quantified by conditional probability $P(M_k \mid \underline{X}_M)$, $k = 1, 2, \dots, N_{Lmax}$. Then, the most probable model class M_k^* is determined by comparing the conditional probabilities $P(M_k \mid \underline{X}_M)$ of all N_{Lmax} candidate model classes and selecting the one with the maximum value of $P(M_k \mid \underline{X}_M)$. The number of soil strata corresponding to M_k^* is taken as the most probable number k^* of soil strata.

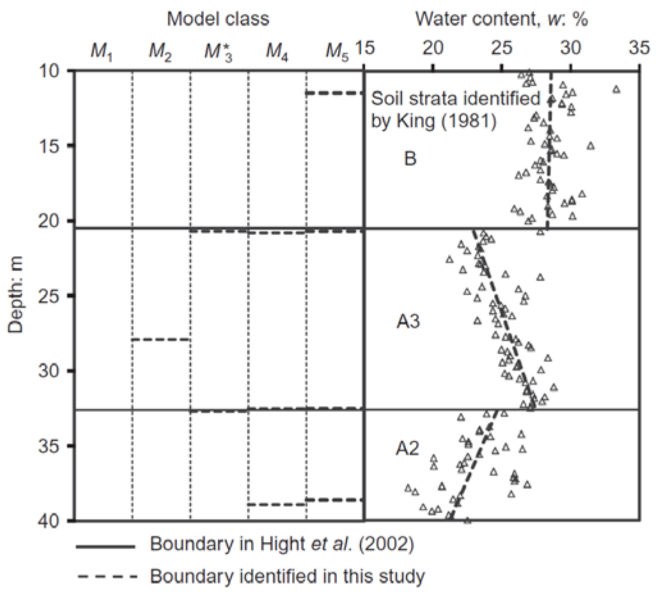

Figure 6-5 Identification of soil strata in London Clay (Wang et al. 2014).

As shown in Figure 6-5, BMCS method has been successfully developed to identify different lithological units of LCF using water content data (Wang et al. 2014). In addition, BMCS method has been used together with CPT data to classify soil behavior types (SBT) and identify soil layers (Wang et al. 2013) or to identify statistically homogeneous soil layers and their associated spatial variability parameters in each statistically homogeneous soil layer (Cao and Wang 2013).

6.4.6 Other applications

Embankments. In one of the earliest applications of Bayesian analysis to geotechnical problems, Honjo et al. (1994) present the case study of an embankment on soft clay to illustrate the effectiveness of what they call a "new type of indirect inverse analysis procedure", which they call extended Bayesian method, EBM. The proposed method is based on a Bayesian model proposed by Akaike (1978) and combines objective information and subjective information. Calle et al. (2005) apply a Bayesian updating concept to develop a method to predict expected mean values and standard deviations of embankment settlement, as function of time. The method is based on both prior assumptions regarding expected means and standard deviations of settlement parameters and computation model uncertainty, as well as actually observed settlement behavior, e.g. during the construction stage. Huang et al. (2014) presents two examples where Bayesian statistical methods can be used for the prediction of future performance. The second example is to update embankment settlement predictions when field settlement monitoring data are available. More recently, Kelly & Huang (2015) present a proof of concept study to assess the potential for Bayesian updating to be combined with the observational method to allow timely and accurate decision-making during construction of embankments on soft soils.

Tunnels. Lee & Kim (1999) adopt EBM for a finite element analysis implemented to predict the ground response. In particular, they determine various geotechnical parameters of a FE implementation of an actual tunnel site in Pusan, Korea, including the elastic modulus, the initial horizontal stress coefficient at rest, the cohesion and the internal friction angle. Cho et al. (2006) combine EBM with a 3-dimensional finite element analysis to predict ground motion by using relative convergence as observation data. The proposed back-analysis technique is applied and validated by using the measured data from two tunnel sites in Korea. Camos et al. (2016) present a Bayesian method for updating the predicted tunneling-induced settlements when measurements are available. They also show how maximum allowable settlements, which are used as threshold values for monitoring of the construction process, can be determined based on reliability-based criteria in combination with measurements. The proposed methodology is applied to a group of masonry buildings affected by the construction of a metro line tunnel in Barcelona, Spain.

Piles. Goh et al. (2005) used a Bayesian neural network algorithm to model the relationship between the soil undrained shear strength, the effective overburden stress, and the undrained side resistance alpha factor for drilled shafts. The proposed approach provides information on the characteristic error of the prediction that arises from the uncertainty associated with interpolating noisy data. Kerstens (2006) deals with the prediction of the ultimate limit state (bearing capacity) of a single foundation pile. The proposed Bayesian statistical method, which combines information on pile capacity with the results of full scale tests, is applied to establish the probability of contending pdf's of the model uncertainty. Huang et al. (2014) presents two examples where Bayesian statistical methods can be used for the prediction of future performance. The first example is to update the capacity of piles using load test results.

Deep excavations. The back analysis or inverse analysis of the field instrumentation data is a common technique to ascertain the design parameter validity in deep excavation projects. Canavate-Grimal et al. (2015) propose a Bayesian-type methodology to solve inverse problems which relies on the reduction of the numerical cost of the forward simulation through stochastic spectral surrogate models. The proposed methodology is validated with three calibration examples.

An in-depth discussion on how the Bayesian methods can be applied in different geotechnical problems can be found in Baecher (2017).

6.5 Possible/Suggested application in relation to Eurocodes

The Eurocodes consist of a series of ten European Standards that provide common approaches for the design of several civil engineering problems. Among the ten standards, the Eurocode 7: Geotechnical design (EN 1997) documents how to design geotechnical structures using the limit state design. This code covers a few important geotechnical design aspects including the basis of geotechnical design, geotechnical data, spread foundations, pile foundations, anchorages, retaining structures, hydraulic failure, overall stability and embankments. Eurocode 7 adopts the partial factors in geotechnical design, which is the Level I reliability-based design: semi-probabilistic approaches. This code was approved by the European Committee for Standardization (CEN) in 2006 and has been mandatory in member countries since 2010.

In Eurocode 7, it is noted that the observational method is recommended in the geotechnical design when the prediction of geotechnical behavior is difficult. This reflects an acknowledgment from Eurocode 7 of the challenges associated with lack of knowledge and uncertainties that are inherent in geotechnical model predictions and the dilemma associated with the selection of design soil properties. Along these lines, and based on the aforementioned review of Bayesian approaches, the following suggested applications are recommended in relation to Eurocodes:

(1) **Bayesian calibration of parameter uncertainty for enhancing design reliability.** The utilization of Bayesian techniques to characterize the uncertainty in design parameters is critical for reliability assessments using the geotechnical prediction models that are embedded in the Eurocode. Mathematically, the updated soil parameters in the presence of site-specific data can be expressed as the product of a constant term, prior distribution of soil parameter, and the likelihood of observing the data (Equation 1). The updated soil parameters can better represent the site-specific variability. Example applications include but are not limited to shallow and deep foundations, slope stability and retaining structures. This application of Bayesian approaches is recommended to combine Eurocodes with local field load tests.

(2) **Bayesian calibration of model uncertainty/bias for enhancing design reliability.** The design equations documented in Eurocodes provide general procedures for various designs across Europe. The local experience can dominate the use of Eurocodes. Thus, its regional and site-specific applications involve uncertainties due to model error or model bias. This may cause either the overestimation or underestimation of the design reliability index. In this regard, it is advisable to conduct the Bayesian calibration of the design models in Eurocodes using the field load test or inspection database. Depending on the scale of the database used, the in-site or the cross-site model error/bias can be obtained. This work can significantly enhance the local suitability of Eurocodes.

(3) **Bayesian calibration of partial factors for consistent reliability level.** Eurocode 7 adopts partial factors in the limit state design and the partial factors are originally calibrated based on the existing database and engineering judgement. Due to the limitations of the calibration database used, a consistent reliability level may not be achieved for a specific design due to local site variability. Therefore, it is recommended that Bayesian calibration of the partial factors can be conducted to achieve consistent reliability levels (e.g., Ching et al. 2013; Juang et al. 2013).

(4) **Use of Bayesian statistics and prior knowledge for selection of characteristic values for soil or rock properties**. In many cases, determination of the characteristic values of geotechnical parameters is a key step for application of Eurocode 7. In geotechnical engineering, a transformation model that relates the design soil parameter to the site investigation result (e.g., SPT N versus ϕ') is typically established by regional data or general data in the literature, which can serve as "prior" information for correlation behaviors among various soil parameters. Given the site-specific measurement (e.g., SPT N), one can adopt the Bayesian method to obtain a more accurate PDF of a soil parameter (e.g., ϕ') than that estimated directly based on transformation model, which can then be used to derive a point estimate as well as 95% confidence interval for the soil parameter. Details of the above technique

are described in a separate report for the discussion group entitled "*Transformation models and multivariate soil databases*".

On the other hand, Bayesian equivalent sample method has been developed to integrate prior knowledge with the often limited site-specific measurement data and transform the integrated knowledge into a large number of equivalent samples using MCMC (Wang and Cao 2013; Cao and Wang 2014; Wang and Aladejare 2015; Wang et al. 2016a). Excel-based user-friendly software has also been developed for the Bayesian equivalent sample method (Wang et al. 2016b). Details of the method, software and application examples are referred to a separate report for the discussion group entitled "*Selection of characteristic values for rock and soil properties using Bayesian statistics and prior knowledge*".

6.6 Challenges and limitations

Despite the usefulness of the Bayesian methods as described above, the real application of the Bayesian method in geotechnical engineering is quite limited. Several possible causes are identified for such a dilemma, which may provide future directions for better use of the Bayesian methods.

First, most engineers are not fully aware of the benefits of Bayesian methods. As a result, there is a lack of willingness for common engineers to use the Bayesian method in practice. On the other hand, the research on the application of Bayesian methods in geotechnical engineering is quite active. To narrow such a gap, researchers in Bayesian geotechnics are encouraged to outreach to the industry to improve the communication with the practicing profession.

Second, most geotechnical engineers do not have special training in Bayesian statistics, which indeed requires advanced statistical concepts and in some cases knowledge of programming. This challenge may be tackled from two directions. Since reliability-based design courses have been incorporated into the curriculum of civil engineering programs in many institutions, instructors in these courses may incorporate Bayesian statistics as a main topic that is covered in the course. Second, TC304 may also consider organizing short courses on Bayesian statistics at different occasions to help interested geotechnical engineers to develop knowledge and capability to solve geotechnical problems using the Bayesian method.

Third, in many occasions, the application of Bayesian methods involves quite some computational effort wherein specialized software may be needed. Currently, however, very few geotechnical engineering software has such capability. Researchers in the Bayesian geotechnics may consider developing easy-to-use procedures for implementing the Bayesian method for geotechnical applications. For instance, The Solver in Excel is a powerful and convenient optimization tool which may significantly facilitate the application of Bayesian methods in geotechnical engineering.

Last but not least, how to specify the prior information could sometimes be challenging. In this respect, efforts can be conducted to recommend rational/practical practices on how to derive the prior information in a more objective and more defensive way. For instance, the regional experience is often one important source of information for deriving the prior distribution. Recent studies have shown that the prior information can be derived quantitatively by using a multi-level Bayesian modeling approach. The calibration of prior information may significantly facilitate the application of Bayesian methods in geotechnical engineering.

6.7 Conclusions

In this report, a systematic investigation has been conducted on the usefulness of the Bayesian method in geotechnical engineering. The investigation showed that the Bayesian method may have a wide range of applications whenever information combination is needed. Nevertheless, a knowledge gap is observed to exist between the academic research and practical application. Recommendations are made to leverage the power of the Bayesian method into practice.

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